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THE ENERGY-MOMENTUM TENSOR FOR THE GRAVITATIONAL FIELD.

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The field theoretical description of the general relativity (GR) is further developed. The action for the gravitational field and its sources is given explicitely. The equations of motion and the energy-momentum tensor for the gravitational field are derived by applying the variational principle. We have succeeded in constructing the unique gravitational energy-momentum tensor which is 1) symmetric, 2) conserved due to the field equations, and 3) contains not higher than the first order derivatives of the field variables. It is shown that the Landau-Lifshitz pseudotensor is an object most closely related to the derived energy-momentum tensor.

1. Introduction

For any Lagrangian based field theory a 'metrical' energy-momentum tensor is defined by

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta L}{\delta g_{\mu\nu}}, \tag{1}$$

where $g_{\mu\nu}$ is the metric tensor, L is the Lagrangian density of the field and g is the determinant of $g_{\mu\nu}$. The field equations (the Euler-Lagrange equations) are derived by applying the variational principle:

$$\frac{\delta L}{\delta \phi_A} = 0, \tag{2}$$

where ϕ_A are field variables. In the geometrical description of GR, one identifies the gravitational field with the geometry of the curved space-time, and the metric tensor $g_{\mu\nu}$ plays

the dual role. On one side, the metric tensor $g_{\mu\nu}$ defines the metrical relations in the space-time, on the other side, the components of the $g_{\mu\nu}$ are regarded as the gravitational field variables. The variation of L with respect to $g_{\mu\nu}$ gives the Einstein equations. If one defines the gravitational energy-momentum tensor according to (1), it becomes equal to zero due to the Einstein field equations.

The quantities which are normally being used are various energy-momentum pseudotensors. They are known to be unsatisfactory. The lack of a rigorously defined energy-momentum tensor for the gravitational field becomes especially acute in problems such as quantization of cosmological perturbations. We believe that the difficulty lies in the way we treat the gravitational field, not in the nature of gravity as such.

The Einstein gravity can be perfectly well formulated as a field theory in the Minkowski space-time. This allows us to introduce a perfectly acceptable energy-momentum tensor for the gravitational field.

2. The field theoretical approach to gravity

The field theoretical approach treats gravity as a nonlinear tensor field $h^{\mu\nu}$ given in the Minkowski space-time. This approach has a long and fruitful history (see for example[1–3]). However, the previous work was focused on demonstrating that the Einstein equations are, in a sense, most natural and unavoidable, rather than on any practical applications of this approach.

We will follow a concrete representation developed in[3]. The gravitational field is denoted by $h^{\mu\nu}$, and the metric tensor of the Minkowski space-time by $\gamma^{\mu\nu}$. The Christoffel symbols associated with $\gamma^{\mu\nu}$ are denoted by $C^{\tau}{}_{\mu\nu}$, the covariant derivatives with respect to $\gamma^{\mu\nu}$ by ";" and the curvature tensor of the Minkowski space-time is

$$\breve{R}_{\alpha\beta\mu\nu}(\gamma^{\rho\sigma}) = 0. \tag{3}$$

A. The variational principle

The action for the gravitational field is

$$S^g = -\frac{1}{2c\kappa}\int L^g \; d^4x, \tag{4}$$

where $\kappa = 8\pi G/c^4$.

The energy-momentum tensor for the gravitational field is defined in the traditional manner, as the variational derivative with respect to the metric tensor $\gamma_{\mu\nu}$:

$$\kappa t^{\mu\nu}|_v = -\frac{1}{\sqrt{-\gamma}}\frac{\delta L^g}{\delta \gamma_{\mu\nu}}, \tag{5}$$

The field equations are derived by applying the variational principle with respect to the field variables $h^{\mu\nu}$ (gravitational potentials). It is convenient (but not necessarily) to consider the generalised momenta $P^{\alpha}_{\;\mu\nu}$, canonically conjugated to the generalised coordinates $h^{\mu\nu}$, as independent variables . This is an element of the Hamiltonian formalism, which is known also as the first order variational formalism. So, the field equations, in the framework of the first order formalism, are

$$\frac{\delta L^g}{\delta h^{\mu\nu}} = 0, \tag{6}$$

$$\frac{\delta L^g}{\delta P^{\tau}_{\;\mu\nu}} = 0. \tag{7}$$

The concrete Lagrangian density for the gravitational field used in[3] is given by:

$$L^g = \sqrt{-\gamma}\left(h^{\rho\sigma}{}_{;\alpha}P^{\alpha}_{\;\rho\sigma} - \frac{1}{2}\Omega^{\rho\sigma\alpha\beta}{}_{\omega\tau}P^{\tau}_{\;\rho\sigma}P^{\omega}_{\;\alpha\beta}\right), \tag{8}$$

where

$$\begin{aligned}\Omega^{\rho\sigma\alpha\beta}{}_{\omega\tau} \equiv \frac{1}{2}[&(\gamma^{\rho\alpha}+h^{\rho\sigma})Y^{\sigma\beta}{}_{\omega\tau} + (\gamma^{\sigma\alpha}+h^{\sigma\alpha})Y^{\rho\beta}{}_{\omega\tau} + \\ &(\gamma^{\rho\beta}+h^{\rho\beta})Y^{\sigma\alpha}{}_{\omega\tau} + (\gamma^{\rho\alpha}+h^{\rho\alpha})Y^{\sigma\beta}{}_{\omega\tau}]\end{aligned} \tag{9}$$

and

$$Y^{\rho\alpha}{}_{\sigma\beta} \equiv \delta^{\rho}_{\sigma}\delta^{\alpha}_{\beta} - \frac{1}{3}\delta^{\rho}_{\beta}\delta^{\alpha}_{\sigma}.$$

B. The field equations

By direct calculation of the variational derivatives of the Lagrangian density (8) one can obtain the field equations:

$$\frac{1}{\sqrt{-\gamma}}\frac{\delta L^g}{\delta h^{\mu\nu}} \equiv -\left(P^{\alpha}_{\ \mu\nu;\alpha} + P^{\alpha}_{\ \mu\beta}P^{\beta}_{\ \nu\alpha} - \frac{1}{3}P_{\mu}P_{\nu}\right) = 0, \tag{10}$$

$$\frac{1}{\sqrt{-\gamma}}\frac{\delta L^g}{\delta P^{\tau}_{\ \mu\nu}} \equiv h^{\mu\nu}{}_{;\tau} - \Omega^{\mu\nu\alpha\beta}{}_{\omega\tau}P^{\omega}_{\ \alpha\beta} = 0, \tag{11}$$

where $P_{\rho} \equiv P^{\alpha}_{\ \rho\alpha}$. Equation (11) provides the link between $P^{\alpha}_{\ \mu\nu}$ and $h^{\mu\nu}$:

$$h^{\mu\nu}{}_{;\tau} = \Omega^{\alpha\beta\mu\nu}{}_{\tau\omega}P^{\omega}_{\ \alpha\beta}, \tag{12}$$

One can also resolve equation (12) in terms of $P^{\tau}_{\ \mu\nu}$:

$$P^{\tau}_{\ \mu\nu} = \Omega^{-1}_{\rho\sigma\mu\nu}{}^{\tau\omega}h^{\rho\sigma}{}_{;\omega}, \tag{13}$$

where $\Omega^{-1}_{\rho\sigma\mu\nu}{}^{\tau\omega}$ is the inverse matrix to the matrix $\Omega^{\alpha\beta\mu\nu}{}_{\tau\omega}$, namely

$$\Omega^{\mu\nu\alpha\beta}{}_{\omega\tau}\Omega^{-1}_{\rho\sigma\mu\nu}{}^{\tau\psi} \equiv \frac{1}{2}\delta^{\psi}_{\omega}(\delta^{\alpha}_{\rho}\delta^{\beta}_{\sigma} + \delta^{\alpha}_{\sigma}\delta^{\beta}_{\rho}). \tag{14}$$

The field equations (10) could be also derived using the Lagrangian formalism, known also as a second order variational formalism. To implement this one has to consider $P^{\alpha}_{\ \mu\nu}$ as known functions of $h^{\mu\nu}$ and $h^{\mu\nu}{}_{;\alpha}$ (see (13)) and substitute $P^{\tau}_{\ \mu\nu}$ into the Lagrangian (8). After performing this substitution, the Lagrangian takes the following form

$$L^g = \frac{1}{2}\sqrt{-\gamma}\Omega^{-1}_{\rho\sigma\alpha\beta}{}^{\omega\tau}h^{\rho\sigma}{}_{;\tau}h^{\alpha\beta}{}_{;\omega}, \tag{15}$$

which is explicitely quadratic in term of "velocities" $h^{\mu\nu}{}_{;\tau}$. The field equations, in framework of the second order variational formalism, are

$$\frac{\delta L^g}{\delta h^{\mu\nu}} = 0 \tag{16}$$

These equations are exactly the same as equations (10), if one takes into account the link (13).

C. Connection to the geometrical GR

Equations (10), (11) are fully equivalent to the Einstein equations in the geometrical approach. To demonstrate the equivalence, one has to introduce new quantities $g^{\mu\nu}$ according to the definition:

$$\sqrt{-g}g^{\mu\nu} = \sqrt{-\gamma}(\gamma^{\mu\nu} + h^{\mu\nu}) \tag{17}$$

and interpret $g_{\mu\nu}$ as the metric tensor of a curved space-time. The matrix $g_{\mu\nu}$ is the inverse matrix to $g^{\mu\nu}$

$$g_{\mu\alpha}g^{\nu\alpha} = \delta^{\nu}_{\mu} \tag{18}$$

and g is the determinant of $g_{\mu\nu}$.

The Christoffel symbols $\Gamma^{\alpha}{}_{\mu\nu}$ constructed from $g_{\mu\nu}$, with (13) and (17) taken into account, have the following form

$$\Gamma^{\alpha}{}_{\mu\nu} = C^{\alpha}{}_{\mu\nu} - P^{\alpha}{}_{\mu\nu} + \frac{1}{3}\delta^{\alpha}_{\mu}P_{\nu} + \frac{1}{3}\delta^{\alpha}_{\nu}P_{\mu}. \tag{19}$$

The vacuum Einstein's equations

$$R_{\mu\nu} = \Gamma^{\alpha}{}_{\mu\nu,\alpha} - \frac{1}{2}\Gamma_{\mu,\nu} - \frac{1}{2}\Gamma_{\nu,\mu} + \Gamma^{\alpha}{}_{\mu\nu}\Gamma_{\alpha} - \Gamma^{\alpha}{}_{\mu\beta}\Gamma^{\beta}{}_{\nu\alpha} = 0, \tag{20}$$

with $\Gamma^{\alpha}{}_{\mu\nu}$ taken from (19), reduce to

$$R_{\mu\nu} = \breve{R}_{\mu\nu} - \left(P^{\alpha}{}_{\mu\nu;\alpha} + P^{\alpha}{}_{\mu\beta}P^{\beta}{}_{\nu\alpha} - \frac{1}{3}P_{\mu}P_{\nu}\right) = 0, \tag{21}$$

which are exactly the field equations (10), because $\breve{R}_{\mu\nu} = 0$.

3. The energy-momentum tensor for the gravitational field

In general, the energy-momentum tensor derived from the Lagrangian (8) according to the definition (5) contains the second order derivatives of the gravitational potentials $h^{\mu\nu}$:

$$\kappa t^{\mu\nu}|_v = \frac{1}{2}\gamma^{\mu\nu} h^{\rho\sigma}{}_{;\alpha} P^{\alpha}_{\ \rho\sigma} + [\gamma^{\mu\rho}\gamma^{\nu\sigma} - \frac{1}{2}\gamma^{\mu\nu}(\gamma^{\rho\sigma} + h^{\rho\sigma})](P^{\alpha}_{\ \rho\beta} P^{\beta}_{\ \sigma\alpha} - \frac{1}{3} P_\rho P_\sigma) + Q^{\mu\nu}, \tag{22}$$

where

$$Q^{\mu\nu} = \frac{1}{2}(\delta^\mu_\rho \delta^\nu_\sigma + \delta^\nu_\rho \delta^\mu_\sigma)[-\gamma^{\rho\alpha} h^{\beta\sigma} P^{\tau}_{\ \alpha\beta} + (\gamma^{\alpha\tau} h^{\beta\rho} - \gamma^{\alpha\rho} h^{\beta\tau}) P^{\sigma}_{\ \alpha\beta}]_{;\tau}. \tag{23}$$

Some of the second order derivatives (but not all) can be excluded by using the field equations. We regard an energy-momentum tensor physically satisfactory if it does not depend on the second derivatives of the field potentials. The remaining freedom in the Lagrangian (8), which preservs the field equations (10), (11), allows us to build such an object.

Our aim is to construct a symmetric conserved energy-momentum tensor which does not contain the higher than the first order derivatives of the gravitational potentials $h^{\mu\nu}$. These are the demands which should be met by a fully acceptable energy-momentum tensor.

To derive such an energy-momentum tensor we use the most general Lagrangian density consistent with the Einstein equations (10), (11) and with the variational procedure:

$$L^g = \sqrt{-\gamma}\left[h^{\rho\sigma}{}_{;\alpha} P^{\alpha}_{\ \rho\sigma} - \frac{1}{2}\Omega^{\rho\sigma\alpha\beta}{}_{\omega\tau} P^{\tau}_{\ \rho\sigma} P^{\omega}_{\ \alpha\beta}\right] + \sqrt{-\gamma}\Lambda^{\alpha\beta\rho\sigma} \breve{R}_{\alpha\rho\beta\sigma}, \tag{24}$$

where $\breve{R}_{\alpha\rho\beta\sigma}$ is the curvature tensor (3) constructed from $\gamma_{\mu\nu}$. We have added zero to the original Lagrangian, but this is a typical way of incorporating a constraint (in our case, $\breve{R}_{\alpha\rho\beta\sigma} = 0$) by means of the undetermined Lagrange multipliers. The multipliers $\Lambda^{\alpha\mu\beta\nu}$ form a tensor which depends on $\gamma^{\mu\nu}$ and $h^{\mu\nu}$. The added term affects the energy-momentum tensor, but does not change the field equations.

The field equations are derived from (24) using the variational principle as it was described above (see (6), (7)). The equations derived from the new Lagrangian are exactly the same as original equations (10), (11).

The energy-momentum tensor directly derived from (24) according to (5) is now modified as compared with (22):

$$
\begin{aligned}
\kappa t^{\mu\nu}|_v = {} & \frac{1}{2}\gamma^{\mu\nu}h^{\rho\sigma}{}_{;\alpha}P^{\alpha}_{\rho\sigma} + [\gamma^{\mu\rho}\gamma^{\nu\sigma} - \frac{1}{2}\gamma^{\mu\nu}(\gamma^{\rho\sigma}+h^{\rho\sigma})](P^{\alpha}{}_{\rho\beta}P^{\beta}{}_{\sigma\alpha} - \\
& \frac{1}{3}P_\rho P_\sigma) + Q^{\mu\nu} + \frac{1}{\sqrt{-\gamma}}\frac{\partial[\sqrt{-\gamma}\gamma_{\alpha\tau}\Lambda^{\tau\beta\rho\sigma}]}{\partial\gamma_{\mu\nu}}\breve{R}^{\alpha}{}_{\rho\beta\sigma} - \\
& (\Lambda^{\nu\beta\mu\alpha} + \Lambda^{\mu\beta\nu\alpha})_{;\alpha;\beta}.
\end{aligned} \tag{25}
$$

This tensor still contains the second derivatives of the gravitational potentials $h^{\mu\nu}$, but the multipliers $\Lambda^{\alpha\beta\rho\sigma}$ can be chosen in such a way that the remaining terms which contain the second derivatives of $h^{\mu\nu}$ (and which could not be excluded by using the field equations) can now be removed. We have shown that the unique choice of $\Lambda^{\alpha\beta\rho\sigma}$ is

$$
\Lambda^{\alpha\beta\rho\sigma} = -\frac{1}{4}(h^{\alpha\beta}h^{\rho\sigma} - h^{\alpha\sigma}h^{\beta\rho}). \tag{26}
$$

As a result we have obtained the energy-momentum tensor (which we call the true energy-momentum tensor), which is

- 1) symmetric,
- 2) conserved, $t^{\mu\nu}{}_{;\nu} = 0$, as soon as the field equations are satisfied,
- 3) does not contain any derivatives of $h^{\mu\nu}$ higher than the first order.

It is necessarily to emphasize that $t^{\mu\nu}$ is a tensor (not a pseudotensor), so that it transforms as a tensor under arbitrary coordinate transformations.

The true energy-momentum tensor has the following final form:

$$
\begin{aligned}
\kappa t^{\mu\nu} = {} & \frac{1}{4}[2h^{\mu\nu}{}_{;\rho}h^{\rho\sigma}{}_{;\sigma} - 2h^{\mu\alpha}{}_{;\alpha}h^{\nu\beta}{}_{;\beta} + 2g^{\rho\sigma}g_{\alpha\beta}h^{\nu\beta}{}_{;\sigma}h^{\mu\alpha}{}_{;\rho} + \\
& g^{\mu\nu}g_{\alpha\rho}h^{\alpha\beta}{}_{;\sigma}h^{\rho\sigma}{}_{;\beta} - 2g^{\mu\alpha}g_{\beta\rho}h^{\nu\beta}{}_{;\sigma}h^{\rho\sigma}{}_{;\alpha} - \\
& 2g^{\nu\alpha}g_{\beta\rho}h^{\mu\beta}{}_{;\sigma}h^{\rho\sigma}{}_{;\alpha} + \frac{1}{4}(2g^{\mu\delta}g^{\nu\omega} - g^{\mu\nu}g^{\omega\delta})(2g_{\rho\alpha}g_{\sigma\beta} - \\
& g_{\alpha\beta}g_{\rho\sigma})h^{\rho\sigma}{}_{;\delta}h^{\alpha\beta}{}_{;\omega}],
\end{aligned} \tag{27}
$$

where $g_{\alpha\beta}$ and $g^{\alpha\beta}$ are defined by (17) and (18).

It is important to point out that $t^{\mu\nu}$ can be numerically related with the Landau-Lifshitz pseudotensor[4]. If one introduces the quantities $g^{\mu\nu}$ according to (17) and uses the Lorentzian

coordinate frame of reference ($\gamma_{00} = 1, \gamma_{11} = \gamma_{22} = \gamma_{33} = -1$) one obtains the relationship between $t^{\mu\nu}_{LL}$ (the Landau-Lifshitz pseudotensor) and $t^{\mu\nu}$:

$$t^{\mu\nu} = (-g)t^{\mu\nu}_{LL}. \tag{28}$$

4. Gravitational field with matter sources

So far we were working with gravitational equations without matter sources, but all procedure cosidered above is also true for gravitational field in presence of matter. In order to maintain equivalence with the Einstein equations in its geometrical form one needs to ensure the universal coupling of gravity to other physical fields (matter sources). The universal coupling between gravity and matter is presented as the following functional dependence of the matter Lagrangian density:

$$L^m = L^m(\sqrt{-\gamma}(\gamma^{\mu\nu} + h^{\mu\nu}); \phi_A; \phi_{A,\alpha}), \tag{29}$$

where ϕ_A is an arbitrary matter field. Then, the energy-momentum tensor $\tau^{\mu\nu}$ of matter sources and their interaction with gravity (derived from L^m according to the same rule (5)) participates on equal footing with $t^{\mu\nu}$ as the right hand side of the field equations:

$$\begin{aligned}&[(\gamma^{\mu\nu} + h^{\mu\nu})(\gamma^{\alpha\beta} + h^{\alpha\beta}) - (\gamma^{\mu\alpha} + h^{\mu\alpha})(\gamma^{\nu\beta} + h^{\nu\beta})]_{;\alpha;\beta} = \\ &\qquad = \frac{16\pi G}{c^4}[t^{\mu\nu} + \tau^{\mu\nu}].\end{aligned} \tag{30}$$

These equations are totally equivalent to the geometrical Einstein's equations

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = \kappa T^{\mu\nu}, \tag{31}$$

where $g^{\mu\nu}$ is defined by (17), $R^{\mu\nu}$ is the Ricci tensor constructed from $g^{\mu\nu}$, and $T^{\mu\nu}$ is derived from the same matter Lagrangian (29) according to the rule (1).

5. Conclusion

The derived energy-momentum tensors $t^{\mu\nu}$ and $\tau^{\mu\nu}$ are fully satisfactory from the physical point of view and should be useful tools in practical applications.

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